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PIECEMEAL SOLUTIONS IN THE PROGRAMMING OF OPTIMAL FLIGHT TRAJECTORIES

By Placido Cicala

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PIECEMEAL SOLUTIONS IN THE PROGRAMMING OF OPTIMAL FLIGHT TRAJECTORIES*

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SUMMARY

The general equations that govern the shape of the optimal flight paths that may be executed in a vertical plane are written out in a form which is especially suited to study of the practical situation wherein the complete trajectory is made up out of piecemeal sections of diverse character. As a rule, real flight paths will be found to consist of such spliced together portions of trajectory-segments, along which special local rules of flight will be in force in order to meet a series of variant conditions, such as are usually imposed, for instance, by placarded rules of operation designed to prevent over-stressing of the structure, excessive engine wear, flame-outs, instabilities in the electronics, etc. The criteria which must be followed in linking together these component pieces making up the total contribution to the final trajectory are set forth in general terms in this paper and then the solutions of some particular cases are offered in illustration of the way the recommended optimization procedure is to be applied in practice.

*"Soluzioni Discontinue nei Problemi di Volo Ottimo." Atti della Accademia delle Scienze di Torino, Vol. 90, June 1956. pp. 533-551.

1. Very rarely in actual practice does a vehicle move on an optimal* flight path which is prescribed over its entire length by the same set of governing relationships and restraints. It is more generally the case that the trajectory will have to be pieced together out of a series of individual sections, each with a distinguishingly different character from that of the next segment. It will be true in many instances, for example, that over a part of the flight path the stick (or other control for the angle-of-attack mechanism) may be required to undergo continuous actuation and likewise the throttle regulating the propulsive device may need to be so operated as to produce a continuously changing thrust. On the other hand, along other stretches of the trajectory, it is a common occurrence to find that either or both of the control levers regulating these components of the aeronautic system have come up against their stops, or, if the actuators have not actually come to the end of their allowable travel, it is often required that they remain fixed at a certain intermediate position in order that some particularly advantageous mode of operation will be continued once this regime of flight has been reached.

In addition, restraints of a more indirect nature are often stipulated to be in force over certain parts of the trajectory. Thus,

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* The method of attack employed herein is based on the Mayer method which requires merely that the quantity to be minimized or maximized need be defined only when writing the boundary conditions. The differential system holds regardless of how this quantity is expressed in terms of the end-values of the variables involved, which, in the present instance, are taken to be: time, horizontal and vertical distance, velocity, mass, and the angle of inclination of the tangent to the trajectory.

it frequently happens that a placarded safe-flight limitation, such as a maximum load factor, will be placed as a ceiling over a whole gamut of multiple combinations of flight conditions which are not to be exceeded. A similar type of generally-stated permissible flight envelope usually is staked out also when it is demanded that certain barriers or minimum objectives are to be surmounted at all costs.

Certain of the questions pertaining to such so-called discontinuous variational problems have been discussed by the author in an earlier note (Reference 1). In particular the equations have been developed in this earlier paper which govern the case wherein one wishes to apply constraints which prescribe how the angle of attack is to vary or which regulate the rate at which fuel is to be consumed. In the present study, however, more general "contingent" conditions are considered, and consequently more basic and generally applicable criteria are set up for bringing about the proper union of the various sub-elements composing the composite trajectory.

In formulation of this more versatile analysis of optimal programming with piecemeal solutions, two important variables are introduced which are related to Lagrange multipliers and which are called simply "index values" in this treatment; one of these variables governs the power plant while the other regulates the angle of attack. These index values are continuously variable along the whole flight path, regardless of the way in which it is pieced together. With the aid of the graphs representing the drag polar of the airplane and the operating characteristics (thrust chart) of the motor, one may proceed to link together the sequential pieces of arc which constitute the totality of the contributions to the final connected trajectory. This welding together of the contributory sections must be done in such a way as not to controvert the Weierstrass conditions, and it is the use of the index-values at the transition points between adjacent pieces of the trajectory which affords the means of guaranteeing that a proper join is going to be produced.

2. The mathematical analysis underlying this extended optimization procedure may be developed by starting with the following simple notation and set of definitions. Let the mass of the airplane be denoted by m. This mass is considered to be concentrated at a point, and the velocity vector $\vec{V}$ which is associated with the motion of this point is agreed to lie entirely within a single vertical (x, z)-plane. The following forces are assumed to be applied to the mass in question: the weight W = mg is aligned with the negatively directed vertical ordinate, i.e., it acts along $-\vec{z}$, while the lift is perpendicular to $\vec{V}$ and the net thrust N = T - D is parallel to $\vec{V}$, where the net thrust is the residual force arising from the inequality of thrust and drag. It is assumed that the acceleration due to gravity is a constant, denoted by g. Furthermore, let the angle that the tangent to the trajectory makes with the horizontal be represented by the symbol ϑ, so that ϑ is the angle between $\vec{V}$ and the x-axis.

Then let it be premised that the drag and thrust are to be known functions of altitude, z, speed, V, and respectively, the lift, L, and fuel consumption (rate of decrease in the mass, m), C; so that one may now write:

$$D = D\ (z, V, L) \tag{1}$$

and $$T = T\ (z, V, C,) \tag{2}$$

The set of graphs which show how the thrust, T, varies as a function of the fuel consumption, C, for given values of z and V, is called the "thrust curve" of the engine, while, likewise, the set of graphs which displays how the drag, D, varies as a function of the lift, L, for given values of z and V, is called the "drag polar" of the aircraft in question.

In order to simplify the notation, let the customary convention also be adopted that a dot over a symbol denotes that the total derivative of the so-indicated quantity with respect to time is meant. Then, in addition, a partial derivative with respect to a certain variable will be denoted by affixing a corresponding subscript to the symbol representing the function in question, except that the subscripts i, j, o, or any numeral will have their usual ordering significance. If the ordering indices i or j appear twice as subscripts in any term, it is to be taken for granted that a summation symbol stands in front of the term, denoting in this shorthand way that the sum must be taken of all such terms with every sequential integer supplied throughout the range of the indicated indices. Finally, it will be found convenient to use affixes of capital letters to designate locations, i.e., the value of the quantity f at the point A will be designated by the scheme f^A, while the difference that exists in a quantity at two different locations may be denoted by means of the evaluation convention employing a vertical bar, i.e., $f\Big|_A^Z$ stands for $f^Z - f^A$.

Now let the notation be further generalized by setting

$$y_0 = t \quad , \quad y_2 = z \quad , \quad y_4 = \vartheta$$
$$y_1 = x \quad , \quad y_3 = V \quad , \quad y_5 = m \qquad (3)$$

and introduce the Lagrangian multipliers $\lambda(t)$ so as to form the functions

$$F = \lambda_j \varphi_j \qquad j \text{ running from 1 to 7} \qquad (4)$$

and $$H = \dot{y}_i F_{\dot{y}_i} \qquad i \text{ running from 1 to 5} \qquad (5)$$

where $$\varphi_1 = \dot{x} - V \cos \vartheta \qquad (6)$$

$$\varphi_2 = \dot{z} - V \sin \vartheta \qquad (7)$$

$$\varphi_3 = \dot{V} + g \sin \vartheta - N/m \qquad (8)$$

$$\varphi_4 = V\dot{\vartheta} + g \cos \vartheta - L/m \qquad (9)$$

$$\varphi_5 = \dot{m} + C \qquad (10)$$

$$\varphi_6 = h\ (C, y_1, \cdots, y_5) \qquad (11)$$

$$\varphi_7 = \ell\ (L, y_1, \cdots, y_5) \qquad (12)$$

The fact that $\varphi_1 = \varphi_2 = \varphi_5 = 0$ is obvious from the definitions of the quantities involved. It may also be readily recognized that $\varphi_3 = \varphi_4 = 0$ because these constitute merely the equations of equilibrium between the forces acting along the tangent to and the perpendicular to the trajectory, all confined in the (x,z)-plane.

Now the conditions expressed by the relations $\varphi_6 = 0$ and $\varphi_7 = 0$ may be thought of as conditions of restraint which may be utilized for prescribing, for example, that the attitude control stick is to remain fixed or that the throttle is to be pegged in one position. If it should happen that these additional contingent conditions are not employed for prescribing restraints in this manner, then the system is said to have two degrees of freedom. On the other hand, if both restraints are imposed, then only the boundary values are left free for choice when determining the sought extremal trajectory.

3. In the Mayer variational method for finding the optimum trajectory, one proceeds to find the functions $y_1, \cdots, y_5$, C, and L as functions of t, under the conditions that $\varphi_1 = \cdots = \varphi_7 = 0$. In doing this it is seen that the pertinent Euler's Equations, $dF_{\dot{y}_i}/dt = F_{y_i}$, for i running from 1 to 5, may be written explicitly as follows:

$$\dot{\lambda}_1 = \lambda_6 h_x + \lambda_7 \ell_x \tag{13}$$

$$\dot{\lambda}_2 = \lambda_3 N_z/m = \lambda_6 h_z + \lambda_7 \ell_z \tag{14}$$

$$\dot{\lambda}_3 + \lambda_1 \cos\vartheta + \lambda_2 \sin\vartheta + \lambda_3 N_V/m - \lambda_4 \dot{\vartheta} = \lambda_6 h_V + \lambda_7 \ell_V \tag{15}$$

$$V(\dot{\lambda}_4 - \lambda_1 \sin\vartheta + \lambda_2 \cos\vartheta) - \lambda_3 g \cos\vartheta + \lambda_4 N/m = \lambda_6 h_\vartheta + \lambda_7 \ell_\vartheta \tag{16}$$

$$\dot{\lambda}_5 - \lambda_3 N/m^2 - \lambda_4 L/m^2 = \lambda_6 h_m + \lambda_7 \ell_m \tag{17}$$

In the case of the quantities C and L the pertinent Euler's Equations are $F_C = F_L = 0$, and thus it follows that

$$\lambda_3 T_C - \lambda_5 m = \lambda_6 m h_C \tag{18}$$

and

$$\lambda_4 - \lambda_3 D_L = \lambda_7 m \ell_L \tag{19}$$

Inasmuch as $\partial F/\partial t = 0$, one recognizes that the first integral for F is H = c where c is a constant, and thus a restraining relation that exists between the variables is found in the form

$$\lambda_1 \dot{x} + \lambda_2 \dot{z} + \lambda_3 \dot{V} + \lambda_4 V\dot{\vartheta} + \lambda_5 \dot{m} = c \tag{20}$$

Now this equation may be used to replace one of the Euler Equations, Eqs. (13) to (17), related to the variable y_i, along any portion of the sought trajectory where $\dot{y}_i$ is not identically zero.

For the problem with two degrees of freedom it is clear that there will be 12 dependent variables, namely, C, L, $y_1, \cdots, y_5, \lambda_1, \cdots, \lambda_5$. These 12 variables are to satisfy the system of differential equations $\varphi_1 = \cdots = \varphi_5 = 0$ and are subject to the dictates of Eqs. (13) through (19) besides. In the event that there is only one degree of freedom, then there is to be added one additional condition, either h = 0 or $\ell = 0$, together with one additional variable λ_6 or λ_7. In those situations where both of the conditions h = 0 and $\ell = 0$ are invoked along any piece of trajectory the differential system $\varphi_1 = \cdots = \varphi_7 = 0$ is adequate for determining all the variables except the Lagrangian multipliers, but the latter may be obtained subsequently by recourse to the set of Eqs. (13) through (19).

Further discussion is given in Reference 1 concerning the way in which the boundary conditions should be applied in arriving at the specific optimum shapes constituting the scallops of trajectory that are being sought, and the reader is referred there for information on such particulars.

4. At the junction between two sections of trajectory* for which different expressions for either h or ℓ are prescribed, the Weierstrass-Erdmann corner conditions must be satisfied. If the symbol Δ is now introduced to denote the jump which occurs in any quantity at such a transition point, it will be required that $\Delta y_i = \Delta \lambda_i = \Delta c = 0$ for i running from 1 to 5.

Therefore, in consequence of the fact that the set of null expressions $\varphi_1 = \cdots = \varphi_5 = 0$ holds, one may deduce that

$$\left.\begin{array}{lcl} \Delta \dot{x} = \Delta \dot{Z} = 0 & & mV\Delta\dot{\vartheta} = \Delta L \\ m\Delta\dot{V} = \Delta N & \text{and} & \Delta\dot{m} = -\Delta C \end{array}\right\} \tag{21}$$

and also from the fact that $\Delta c = 0$ it follows that

$$q_1 \Delta C = \Delta N + q_2 \Delta L \tag{22}$$

where q_1 and q_2 are the "index-values", defined as follows:

$$q_1 = \frac{\lambda_5 m}{\lambda_3} \quad \text{and} \quad q_2 = \frac{\lambda_4}{\lambda_3} \tag{23}$$

Further conditions may be imposed on the values of ΔC and ΔL as a result of taking into account the restraints represented by the equations $\varphi_6 = 0$ and $\varphi_7 = 0$.

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* The curve under consideration here is still called a trajectory even when the group of coordinates involved in its description happens to be other than the customary horizontal and vertical location parameters, x and z.

5. Now the Weierstrass function may be written as

$$E = -F_{\dot{y}_i} \quad \overset{*}{\Delta}\dot{y}_i \tag{24}$$

where $\Delta*$ is used to denote the difference in value between any quantity which pertains to the sought extremal path itself and the corresponding value of the quantity in question when it belongs to any arbitrary admissible virtual path. Inasmuch as $\overset{*}{\Delta} y_i = 0$, and because the differential system $\varphi_i = 0$ (for i running from 1 to 5) must be satisfied, it follows that the same set of relations which were valid in the case of the Δ differences, as given by Eqs. (21), must now also apply when dealing with the $\Delta*$ differences. Consequently, it is required that

$$E = \frac{\lambda_3}{m} \quad (q_1 \cdot \Delta{*}C - q_2 \cdot \Delta{*}L - \Delta{*}N) \tag{25}$$

It is well worth noting at this juncture that an interesting and important geometric aid to understanding of the role played by the index-values is made evident when one examines the implications of the Weierstrass condition that $E \geq 0$. Let P_1 be a typical point, representing the power-plant operating conditions, for the optimum trajectory, on the thrust-curve plots (C, T); likewise let P_2 be the corresponding point, representing the aerodynamic behavior, for the optimum trajectory, on the polar diagrams (L, D). Now for the operating condition for which one requires that $\lambda_3 > 0$ it will be necessary to see to it that the whole thrust curve should lie on the side of negative T with respect to the straight line having a slope given by $dT/dC = q_1$ and passing through the point P_1 on the engine charts. Likewise, it will be necessary to see to it that the drag curve should lie on the side of positive D with respect to the straight line having a slope given by $dD/dL = q_2$ and passing through the point P_2 on the drag polar. Consequently, it is easily appreciated that there has thus been set up two forbidden areas on the engine operating plots and on the drag polars, into which the straight lines with slopes q_1 and q_2, respectively, must not penetrate when the operating characteristics for the trajectory in question are traced out.

Inasmuch as the index-values, q_1 and q_2, are the controlling factors entering the Weierstrass condition, it follows that they will be the guiding factors in building up the whole connected trajectory out of the piecemeal solutions; this is the fundamental concept which is to be exploited in the extension of the usual variational problem to apply now to the pieced together path made up of disjointed sections along which variant sets of restraints are imposed.

If it happens that the condition that $h = 0$ is assumed to hold over the entire trajectory, then $\Delta^*C = 0$, and thus the stipulation pertaining to q_1 and the thrust curve is suppressed. On the other hand, if it happens that the condition that $\ell = 0$ is assumed to hold over the entire trajectory, then $\Delta^*L = 0$, and in this instance the stipulation pertaining to q_2 and the drag polar is no longer in force.

6. Some of the special features of the scalloped trajectories which will be obtained when applying the index-value concept in linking the pieces together into a connected optimum path will be

discussed now first of all in general terms. Let attention be focused to begin with on a piece of optimum trajectory along which the stipulation is imposed that $h = 0$ but where $h_C \neq 0$. Now according to Eq. (18), under this hypothesis and along with the supposition that $\lambda_6 \neq 0$, it then follows that $q_1 \neq T_C$, and thus the straight line with slopes q_1 will cut through the thrust curve in general. This situation may be interpreted as follows: If the constraint embodied in the stipulation $h = 0$ were waived so that the power plant would be free to operate in any way that might be desired rather than according to a prescribed program, it would be found that there exist values of $\Delta{*}C$ for which $E < 0$. In other words, it would be possible to find a mode of motor operation which would be an improvement over the constrained optimum under consideration, by allowing the fuel flow to be varied in an appropriate way.

Such a state of affairs would not necessarily occur, however, if the typical point representing the power plant behavior is located at a cusp point of the thrust curve where two sections a' and a'' of the thrust-curve characteristic join together. In this case, then, if the tangents to the two branches a' and a'' of the operation curve are denoted by T_C' and T_C'', respectively (and since it may be assumed that $T_C'' < T_C'$), and if q_1 is such as to fall between the slopes of the tangents in question, so that $T_C' > q_1 > T_C''$, no improvement in execution of the trajectory would ensue by removing the constraint represented by $h = 0$. As soon as the index value q_1 reaches either of the end-values T_C' or T_C'' though, the $h = 0$

restraint must be relinquished, in order that the necessary conditions on the variables be satisfied by the free system.*

Similar considerations hold in the situation where a typical point for the engine operation happens to fall at one of the terminal points of the thrust curves. This represents the situation when the engine is operating with the throttle up against

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* Note added by author during translation: The special term "contingent conditions" is reserved to have one of the following meanings in this treatment: (a) those conditions of constraint, such as the ones $h = 0$ or $\ell = 0$, which are stated unilaterally, i.e., the statement that either $h \gtrsim 0$ or $\ell \gtrsim 0$ must hold is interpreted as a contingent condition according to this definition, (b) discontinuities in the basic data, such as in the thrust curve cited in the present discussion, or (c) anomalous behavior encountered at any stage in the development of the trajectory which would intervene in the mathematical analysis in such a way as to make the determination of the extremum invalid. For more information on this subject consult Section 1.6 of An Engineering Approach to the Calculus of Variations, by P. Cicala, Casa Editrice Levrotto e Bella, Turin, Italy.

The complete trajectory being sought may be thought of as fabricated out of a series of scallops of trajectory along which each of the contingent conditions may or may not happen to be in effect. In those instances where a contingent condition is not supposed to apply, it is dropped from consideration by having the corresponding multiplier vanish, although the same Weierstrass function still continues to hold at all points. Contingent conditions may be taken into account even when they are expressed implicitly (as discussed in Reference 1), but in the present paper they always will be written out explicitly. In either case, however, one must ignore the contingent conditions when the virtual deviations represented by Δ^* are being evaluated. Further information on this score is also supplied in Section 7.1 of the above-cited text on An Engineering Approach to the Calculus of Variations.

its stop. This state of affairs will persist until the straight line with slope q_1 finally ceases to penetrate into the forbidden area lying below the thrust curve. In more general terms it may be stated that whenever an additional condition acts in the manner of a unilateral constraint (such as, $h \geq 0$), then E must be non-negative for the values of Δ*C which satisfy the stipulated inequality $h \geq 0$ and thus only that branch of the operating curve is used which is not rejected through satisfaction of this inequality. The scallop of trajectory along which these conditions apply ($h \geq 0$) is terminated just as soon as q_1 attains the value of T_C that corresponds to the slope of the admissible branch of the operating curve.

Quite analogous conclusions arise in regard to the treatment applying to the drag polar. It may be observed thus that the kind of piecemeal solutions being considered in this extension of the usual variational problem actually are going to be the ones which occur most frequently in the realistic practice of trajectory determination because the drag and thrust operating-curves are generally confined by bounds beyond which they cannot be validly employed and because it is also found very convenient in most cases to replace the complicated graphs of the operating characteristics by broken lines which have gently varying slopes only over short stretches but which exhibit jumps in direction where they are spliced together.

The joining together into the composite optimum path of the scallops of trajectory along which the successively variant sets of conditions are in force is brought about in accordance with the dictates of Eqs. (21) and (22) and through guidance in avoiding forbidden operating regions afforded by knowledge of the index-values. These index-values do not undergo any change when the transition from scallop to scallop is being made. Consequently, if the operating-characteristic curves have a non-zero curvature near the juncture point, then at the actual location of such a join it must hold true that $\Delta L = 0$, and $\Delta C = 0$, so that thus $\Delta N = 0$ there. Another way of interpreting what takes place at such a juncture point is to observe that, because all points on the thrust curve must lie on the same side of the straight line having the slope q_1, and because this direction does not change while the transition is being negotiated, the representative operating point under consideration cannot undergo any jump in value, except if it possibly turns out that a segment of this straight line just happens to coincide with the operating characteristic curve.

Of course, the same sort of argument applies in the case of the drag polars, and consequently one may conclude that at a junction point of the linked together scallops of trajectory there can be finite changes in the value of N only if the operating curves are composed over part of their extent by pieces of straight line.* When this happens, the solution is liable to be complicated by invoking more than one branch of the singular† extremal trajectory which arises

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* The situation which is encountered when the characteristic operating curve has a bitangent (producing local excursion in the curve by bulging away from a line that is tangent to it at two non-consecutive points) has been treated in the initial note of this method, cited previously as Reference 1. This case reduces to the same circumstances which are met when the operating characteristic curve has straight-line portions.

† The extremal trajectory is said to be singular whenever the second derivatives D_{LL} or T_{CC} vanish.

in consequence of having to deal with straight line portions of the operating curves.

7. In order to illustrate the difference between solutions to the optimum trajectory obtained with free-and-unregulated as contrasted to contingently constrained engine operation, the following example may be examined with profit. Consider the case in which the condition is imposed that $\ell = nW - L > 0$, which may be physically interpreted to mean that one wants the normal load factor always to remain less than the specified red-lined limit of n. No further restrictions are to be invoked so long as the operation proceeds with due regard to this limitation that it must always be vouchsafed that $L/W < n$. Just as soon as it comes about that $L = nW$, then from that time on there will be terms $\lambda_7 \cdot ng$ and $- \lambda_7 \cdot m$ to contend with on the right hand sides of Eqs. (17) and (18), respectively. In consequence of the appearance of these new terms, it follows that the line with slope of q_2 will now penetrate into the forbidden zone lying above the drag curve for those abscissa values for which $L > nW$. The scallop of trajectory for which the relation $L = nW$ is in force begins and ends with the provision that $q_2 = D_L$ at these end-points, i.e., this means that $\lambda_7 = 0$ at these extremities.

Similarly, if it is prescribed that the normal acceleration must not exceed a certain specified limiting value of $n \cdot g$, the mathematical statement of the condition to be satisfied is $\ell = nW - L + W \cos\vartheta \geq 0$, and thus the scallop of trajectory traversed with constant normal acceleration has to begin and end with the proviso that $q_2 = D_L$. The only difference between this situation and the one encountered previously lies merely in the inclusion of additional terms on the right hand side of the Eqs. (16), (17), and (18).

If one wishes to impose a prescribed limitation on the magnitude of the allowable resultant acceleration, the procedure to follow is to require that

$$h = n^2W^2 - (L - W\cos\vartheta)^2 - (N - W\sin\vartheta)^2$$

Inasmuch as h contains L in this application, contrary to what was agreed upon in writing Eq. (11), there will be introduced into the right hand side of Eq. (19) the expression

$$2\lambda_6 m\cdot(1 - \alpha D_L)\ (L - W\cos\vartheta)$$

where $\alpha = (N - W\sin\vartheta) / (L - W\cos\vartheta)$, which stands for the ratio between the tangential and normal (centripetal) components of the acceleration. The scallop of trajectory traversed with constant acceleration, so that $h = 0$, thus will start and end with the proviso $q_2 = D_L$. It also will be true that at the same time one has that $q_1 = T_C$ at these terminal locations.

If the situation is conceived of where a part of such a scallop of trajectory is not only executed under the condition of constant acceleration but, furthermore, is traversed under full-throttle operation of the engine, then the negotiation of the transition regions at the beginning and ends of the part of the flight path which abuts against the adjacent stretches of trajectory along which the thrust is programmed must be brought about under the requirement that

$$q_1 = T_C\ (1 - q_2\alpha)/(1 - D_L\ \alpha)$$

A particularly simple subcase of this type of trajectory is represented by purely vertical flight. In this contingency the terminal points of the scallop of flight path along which a constant-acceleration type of trajectory is being executed (so that

T/m = constant in this instance) will correspond to the locations for which it is established that $q_1 = T_C$.

The discussion of the physical meaning that may be ascribed to the Weierstrass condition in the general case of an optimum trajectory executed under conditions of constant resultant acceleration may be spelled out most conveniently with aid of an (L, N) chart, on which the condition h = 0 is represented by a circle. The locus of all points for which E = 0 is tangent to this circle provided that the control levers are not up against their stops. Obviously, it must be taken for granted that no point of the locus E = 0 is to fall interior to this circle. The terminal conditions for a scallop of trajectory traversed under the condition of constant magnitude for the acceleration, then, may be seen to correspond to the instants when the curve E = 0 coalesces into a single representative operating point on the h = 0 circle.

8. It will be instructive now to work out some illustrative examples. It will be found though that even under quite inelaborate underlying hypotheses concerning the nature of the flight, the resulting pieced-together optimal trajectory can emerge as an ensemble composed of rather numerous scallops. For the first application, the following simple set of governing regulations will be assumed:*

(a) a horizontal path is prescribed for which $\ell = \sin\vartheta = 0$

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* For a bibliography pertaining to this problem consult Reference 2.

(b) the thrust is programmed in such a way that $T = kC$ where k is a constant and likewise the rate of fuel flow is bounded so that $0 \leq C \leq C_1$, where C_1 is a constant, and the maximum thrust is denoted by $T_1 = kC_1$

(c) the maximum range is required to be flown, but the time to do it is no object; while the initial and final values of the mass and velocity are to be assigned.

Because of the stipulation written as condition (a) above, it follows that the right hand sides of Eqs. (13) through (19) are all zero, except for Eq. (16). Now the multipliers λ_2 and λ_7 are defined by means of Eqs. (16) and (14), and, because they do not appear anywhere else, then they may be considered as divorced from the differential system.*

Because of the hypothesis written as condition (b) above, the complete trajectory may contain scallops which correspond to the operating condition that $C = 0$ or that $C = C_1$, and likewise the flight path might even incorporate a singular extremal solution that would be derived from application of the Euler relations in the case where $T = kC$. In regard to this latter alternative it may be remarked that under hypothesis (c) listed above, and by use of Eqs. (19) and (20), one finds that in this case

$$q_1 = k \tag{26}$$

* Nevertheless, the determination of the value of these multipliers is useful to carry out in many instances. It is sufficient to remind oneself, for example, that the difference $\lambda_2^Z - \lambda_2^A$, between the values of λ_2 at the extremities of the flight path represents the derivative with respect to flight height of the horizontal range, $x^Z - x^A$; that is, the distance flown from point A to point Z.

and $$\lambda_3 = mV/D \tag{27}$$

while by reference to Eq. (15) it may be deduced that

$$\dot{\lambda}_3 - \lambda_3 D_V/m + 1 = 0 \tag{28}$$

Upon elimination of λ_3 between these two equations, it will be seen that the extremal solution is represented by the relation $\Omega = 0$, where

$$\Omega = (V - k)D - WVD_L + VD_V k \tag{29}$$

The behavior of this solution may be examined to best advantage by consideration of the "trajectory" traced out in the (V,m) plane. First of all, it may be noted that, along the lines C = 0, which lie parallel to the V axis, Eq. (27) is still going to be operative, and meanwhile the value of λ_5 may be computed by aid of Eq. (17), so that it is found without any real trouble that

$$\frac{V}{D}(q_1 - k)\Big|_P^Q = \int_P^Q \frac{\Omega}{D^2}\, dV \tag{30}$$

Thus this working equation allows one to compute what the index value q_1 has to be at the point Q when its value at another point P, located on the same line, is known.

The beginning step in the procedure to be followed in the case of the lines $C = C_1$ is to integrate the relation $\varphi_3 = 0$ with aid of the knowledge that in this case: $\dot{m} = -C_1$. Subsequently, then, one may compute λ_3 by recourse to Eq. (28) and λ_5 may be obtained from use of Eq. (20), and finally the desired index-value, q_1, may be arrived at, by inserting the values of λ_5 and λ_3 into Eq. (23).*

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* Noted added by author during translation: By performing the evaluations outlined, one arrives at the expression

$$\frac{\eta\,\lambda_3(q_1 - k)}{D}\Big|_P^Q = \int_Q^P \eta\,\omega\, dt$$

(footnote continued on next page)

Negotiation of the corners at the juncture-points between various loops of trajectory is governed in all cases by the rules stated in Article 5.

Diagrammatic treatment of such trajectory constructions is illustrated in Figure 1. Several trajectories are shown for solutions all having point Z as a common terminal. For purposes of this figure, it is assumed that the drag is composed of the parasitic and induced varieties, so that the general law holds that $D = D'$

where
$$D' = AV^2 + Bm^2/V^2 \tag{31}$$
in which A and B represent constants.

It is convenient to introduce the reference values

(Footnote continued from previous page)

where
$$\omega D^2 = (V - c) \quad (D - WD_L + kD_V) - kD$$
and where the functions η and λ_3 may be obtained by evaluation of the integrals
$$\ln \eta \Big|_P^Q = \int_Q^P \frac{D_V}{m} dt \qquad \text{and} \qquad \lambda_3 \eta \Big|_P^Q = \int_Q^P \eta \, dt.$$
In this example the constant c appearing in Eq. (20) now vanishes. By the use of the above-given expressions the integrations can be carried out along any line represented by C = constant. In the case where C = 0, one takes $\eta = D$, and then Eqs. (27) and (30) are readily obtained. The above-given expressions are of real aid in locating the terminal points, i.e., the "corners" of the trajectory which correspond to the locus $\Omega = 0$. For instance, if P and Q are corners corresponding to $q_1 = k$ on the line $C = C_1$, then for the scallop of trajectory running from P to Q one has that $\int_P^Q \eta \omega \, dt = 0$.

V_r and m_r, defined as

$$V_r = k \quad \text{and} \quad m_r = V_r^2 \, (A/B)^{1/2} \tag{32}$$

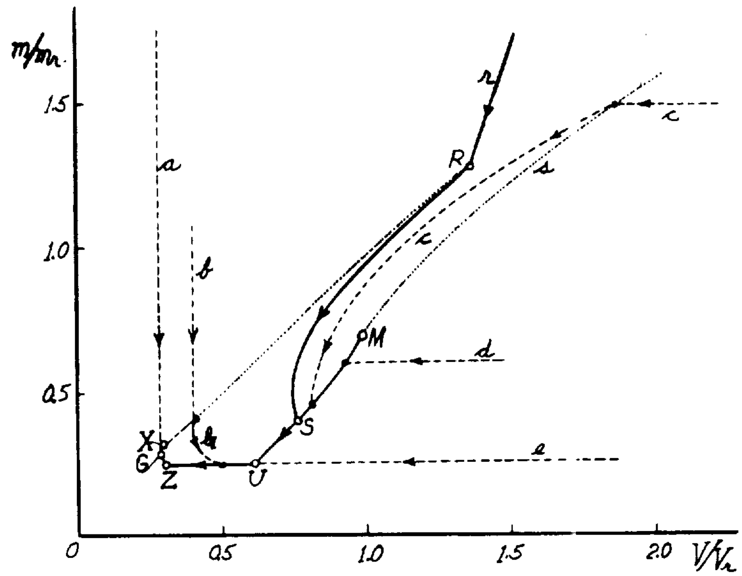

Fig. 1

Optimal Trajectories for Constant Lift and Drag Coefficients

The scallop of trajectory shown as U - M represents part of the extremal solution corresponding to $\Omega = 0$. This piece of trajectory must terminate at the point M, where the fuel flow attains its maximum value, C_1.

If it is conceded that the device under consideration can jettison away a portion of its fuel as an unburned discharge if it is opportune to do so (or, in other words, if the fuel flow C is allowed to reach large values, greater than C_1, while it is still premised that $T = k\,C_1 = T_1$ can only be true) then it will be found that another extremal solution exists. According to Eq. (17), the relationship which defines this new extremal curve, which is characterized by the fact that $T = T_1$ and $q_1 = 0$, is simply

$$T_1 - D + LD_L = 0 \qquad (33)$$

and is represented by the line designated as "r" in Fig. 1.

The integrations needed to trace optimal paths start out from the fixed end-point Z in the treatment being considered here. Some illustrative examples which are pertinent in those cases where the assumed behavior of the thrust curve is not quite so elementary as the relation selected for purposes of the present discussion have been described in some detail in Part II of Reference 2. The developments presented there cover several additional topics of importance, but the following further remarks of general interest are now in order here, in order to provide a clearer physical insight into just what significance can be attached to the various regions of the (V,m)-plane.

In order to gain a deeper appreciation of the factors which govern the construction of the optimal paths under a variety of conditions, then, proceed by considering what kind of path results by tracing out some solutions of a simple variety. For example, consider the following situation. Let the index-value q_1 at the point Z be denoted * by q_1^Z and let it be assumed that its magnitude is such

* Because in the present problem one is dealing with the situation where the extremal trajectory is singular, there will not necessarily be a one-to-one correspondence between an individual initial value and a unique optimal path. For instance, to any value of q_1^Z for which $q_1^Z > \hat{q}$ there will correspond only a single Z-e line, but every trajectory which contains part of the extremal solution $\Omega = 0$ will correspond to the initial value $q_1^Z = \hat{q}$.

that $k < q_1^Z < \hat{q}$,

$$\text{where } \hat{q} = k + \frac{D^2}{V^2} \int_U^Z \frac{\Omega}{D^2}\, dV$$

With these understandings agreed upon it will be seen that then any integration started out by following along the line $C = 0$ through Z will yield the value of $q_1 = k$ at some intermediate point lying between Z and U , and thereafter the integration is carried out along a line $C = C_1$ (such as the line denoted by "b_1" in Fig. 1). Along these lines the index-value q_1 is decreasing. That is to say, in following along the line $C = C_1$ passing through Z the value of q_1 vanishes at X. If the locus of all points where $q_1 = 0$ is drawn in, it is found that this dividing line thus reaches from X and extends upward to the right to intersect the line r at the point R, and its complete shape is sketched in as the upper dotted line in Fig. 1.

The trajectory scallop S-R of a line $C = C_1$ is worth especial attention. Along this arc of curve the index-value q_1 takes on the value zero where it crosses the line "r" (extremal $q_1 = 0$) and it takes on the value k where it crosses the U-M boundary (extremal $q_1 = k$). Beyond the curve R-S, on the lines $C = C_1$ (such as the line labelled "c" in Fig. 1), the index-value, q_1, has the value k at the intersection with the curve S-M, and always remaining positive, it again takes on the value k when the intersection with the corner line M-s is reached. For the branches of the curve $C = C_1$ which cross the line R-r, where q_1 takes on the value zero, the value $q_1 = k$ is attained when the trajectory reaches the line which is the continuation of the M-s curve, constituting the junctures of the lines $C = C_1$ and $C = 0$.

Thus, in summary, the meaning of the various parts of this diagram representing the possible behavior of the variant trajectories may be explained by observing the following:

The terminal-point Z may be reached by starting out at any point which lies in that portion of the (V,m)-plane located above the line Z-e and C = 0, and to the right of the line a-G-Z. Note that the V = constant line labelled "a" is tangent to the curve Z-X at G. This area of permissible starting points is subdivided into three regions of distinctly different significance by means of the corner-line boundaries labelled G-X-R-r, and U-M-s. The region to the right of the line U-M-s is covered with lines for which C = 0 (coasting flight). The region above the boundary G-X-R-r is covered with lines for which V = constant (representing the capability of instant fuel dumping).* The intervening region is covered by optimal trajectories made up out of scallops for which $C = C_1$.

Once the boundaries of these regions have been drawn in on the (V,m)-diagram, then the tracing out of the permissible linked-together trajectories may be carried out quite simply. Starting at any origin A, then, one must begin by following along a trajectory scallop of the sort which is germane to the particular region in which A is located, and this sort of trajectory is followed until one comes to a boundary line. If the crossing of the boundary of the initial region occurs at a point belonging to either of the dotted curves, then one proceeds on and enters into the adjacent region by coursing along the adjoining pertinent scallop. If, however, the

* Note added by author during translation: This situation may be interpreted as meaning that for points located in this region the initial fuel supply would be reduced in order to obtain greater range through reliance on the reduction in induced drag which follows a decrease in weight.

intersection of the initial trajectory scallop occurs at any of the boundaries labelled R-r, or U-M, or Z-U, or G-X, then the optimal path is channelled alone these dividing boundaries themselves.

Any solution which happens to include a piece of the line labelled "r" will have to pass down to the singular solution represented by the line $\Omega = 0$ through means of the "bridge" represented by the line R-S. All paths coming along the scallop U-M will reach Z by passing over the segment U-Z, and those entering the narrow arc G-X will be led into Z over the G-Z channel. The dashed lines depicted in Fig. 1, such as the ones labelled "a", "b", "c", "d", and "e", all represent specific examples of permissible optimal trajectories that end up at Z.

9. As another illustrative example of a piecemeal solution to a trajectory problem, let the same situation be considered again as was just met in the preceding Article, but this time let the assumption concerning the drag characteristics be changed in the following manner. In the present instance, it will be assumed that $D = D'$ provided $V < 0.85\ V_r$, and that $D = 1.8D'$ for $V > 1.05\ V_r$, while in the intermediate velocity interval where $0.85\ V_r < V < 1.05\ V_r$ it will be assumed that $D = (4V - 2.4\ V_r)D'/V_r$. In addition it is taken for granted that $k = 7\ V_r$.

Under these circumstances, then, the trajectories have been calculated by following the method that has been expounded here, and the results have been summarized in the accompanying Figs. 2, 3, and 4. This problem has been selected because of its close bearing on the important case of a jet plane passing through the sonic speed range. Only the final results are considered in interests of brevity.

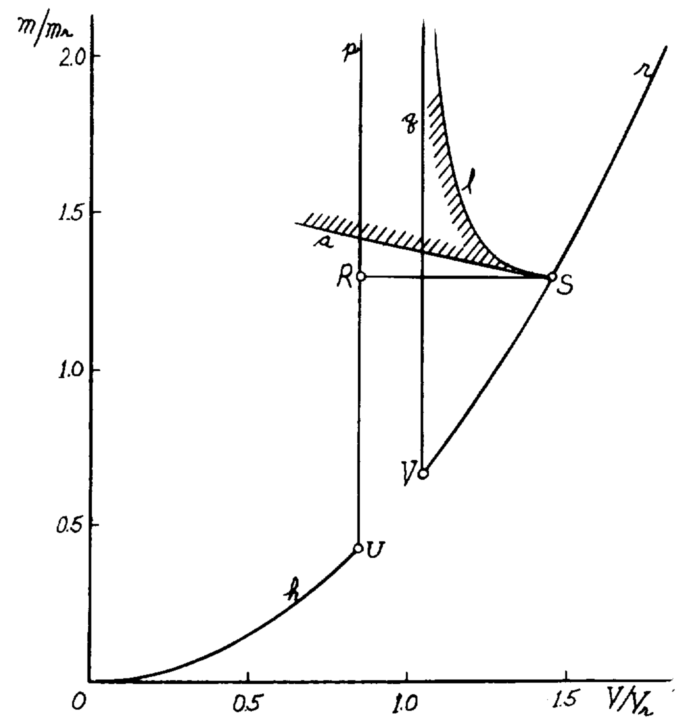

Fig. 2

Optimal Trajectories for Jet Plane Passing Through Sonic Speed Range

With exception of its behavior in the interval $0.85\ V_r < V < 1.05\ V_r$, the extremal solution $\Omega = 0$ does not differ now from what was found to hold in the previous case, so that one may immediately draw in the scallops labelled "h" and "r" in Fig. 2. At the end-points of this transonic interval, denoted by the letters U and V, two vertical lines, "p" and "q", are located; they are parallel to the m-axis, or at least they may be erected as parallel right out of the limits of the diagram.

Another line of importance is the "corner" line denoted by "ℓ"; this curve is the locus of all points where the index-value

q_1 becomes equal to k, according to Eq. (30), by travelling along lines C = 0 after having started out on the line "p", where the value is also k. This line "ℓ" and the upper piece of extremal Ω = 0 which is denoted by "r" will intersect in a point S. The way one traverses this (V,m)-diagram will depend, naturally, on what terminal point is selected for the trajectory as well as on the initial T_1 value.

Some details of permissible trajectories are shown in Figs. 3 and 4, for a portion of the whole field, where two different cases are illustrated for a terminal point lying somewhere below and to the left of the point R, shown in the more inclusive diagram of Fig. 2; the same lettering is used in all these figures (2 through 4). In Fig. 3 the description of what occurs is given for the situation where T_1 happens to be equal to the drag value, D, of a point M lying on the vertical line "p". The case of impulse burning is illustrated by the sketch of Fig. 4.

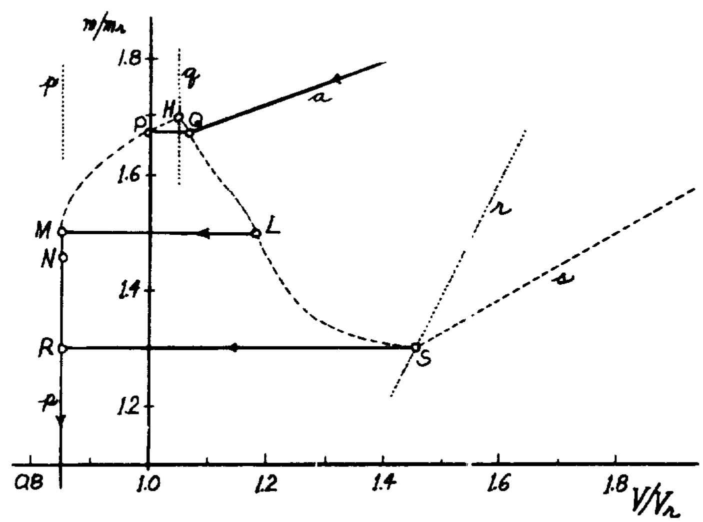

Fig. 3

Construction of Boundaries to Subfields Useful in Composing Piecemeal Optimal Trajectories for Jet Plane Operating in Transonic Range

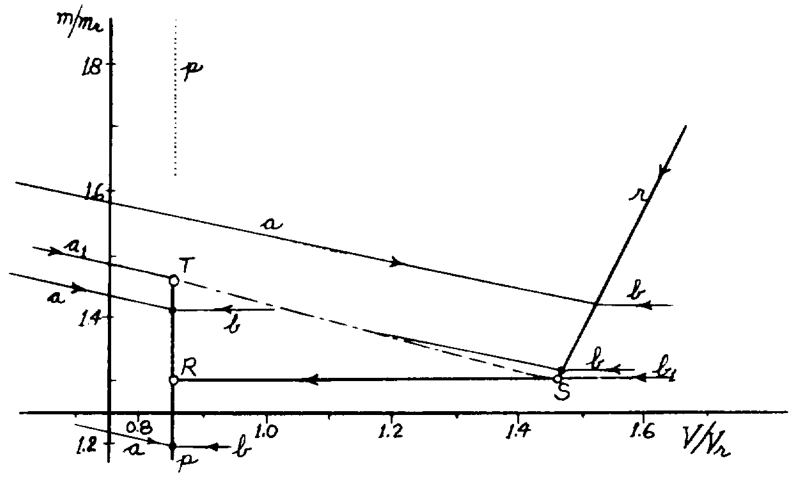

Fig. 4

Portions of Optimal Trajectories for Impulse Burning with Transonic Jet

In the first of these two cases, the line p-M-H-L-S-s represents the boundary between an upper region where $C = C_1$ and the nether region where $C = 0$. The scallop of curve labelled M-H is obtained by determining the points where the index-value q_1 becomes equal to k by starting* from a point of the line "p" located somewhere below the level of M and following the lines $C = C_1$. For instance, starting from the point labelled N where $q_1 = k$, and by travelling along on a curve which passes a little to the left of the dotted line marked M-H, one arrives at a value of k for q_1 once again when the point P is reached (this path has not been drawn in on Fig. 3 because it would be too close to the arc of curve M-H to be readily distinguishable from it).

* The use of the word "starting" merely refers to the sense of the integration, which happens to be that of negative t.

Now if one takes up the integration at points of the M-H scallop, so that by starting with $q_1 = k$ and progressing along the lines $C = 0$ the value of k for q_1 is again attained, the path will end at points such as Q, lying on the piece of boundary curve denoted by H-L. Similarly, if one starts out from a point belonging to the segment of arc M-R, the piece of boundary labelled L-S is obtained, which corresponds to the general boundary curve depicted as line "ℓ" in Fig. 2.

By starting with $q_1 = k$ over any of the boundary S-L-H and from points on the lower portion of line "p", one will find that by travelling over the curves $C = C_1$ the value of $q_1 = k$ will once again be obtained when the line "s" is met. Once the boundaries have been established in this manner, it becomes a simple matter to trace the permissible paths emanating from any initial point.

In regard to the trajectory passing through the point P which comes from an initial point lying to the right of line "s" it may be observed that the linked-together scallops of complete trajectory are composed of: firstly, a segment traced out over the line $C = 0$ rising up to intersect the boundary "s"; secondly, a piece of trajectory traversed over the line $C = C_1$ until the point Q is reached; thirdly, a short segment from Q to P for which $C = 0$; fourthly, the piece of arc running from P to N over which $C = C_1$; and, finally, an extremal arc descending from N along "p" for which $C < C_1$.

In the case of impulse burning, where $C_1 = \infty$, it turns out that one is confronted with the new situation occasioned by the fact that a portion of the trajectory diagram is triply covered; this is the portion of Fig. 2, set off by hatchure lines, which lies between the line labelled "ℓ" and the one labelled "a" (for which $C = \infty$) which is tangent to the line "ℓ". The construction of the trajectory solution for maximum range will thus have to be somewhat more circumspectly dealt with in this eventuality, and, thus, to this end the linking together of the kinds of permissible path* which will be traversed under such circumstances is shown in some detail in Fig. 4. The boundary separating the regions where $C = C_1$ from the regions where $C = 0$ has been shown by the contour labelled p-T-S-r. All the permissible paths having their origins located above the boundary a_1-T-S-b_1 will eventually come together to course along down the extremal solution denoted by "r". As these trajectories are followed further, it is seen that they will reach the line "p" by passing over the "bridge", denoted as the line R-S, and then they proceed on down the extremal "p" itself. If the initial point of the trajectory happens to be located at any point lying below this contour a_1-T-S-b_1, the permissible trajectory is simply traversed directly through the intersection point lying on "p".†

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* The scallops of trajectory corresponding to paths over which $C = C_1$ have been designated by the symbol "a" in Fig. 4, while the paths over which $C = 0$ have been indicated as the family of curves "b" in this figure.

† Nothing about the procedure now being enunciated needs to be changed in any essential way if one wishes to include the possibility that some ballast or cargo is suddenly cast away. This situation may be handled as follows. Let the dropped mass be denoted by m". Let it be assumed that no velocity change takes place when the change from mass m' to the residual mass m' - m" takes place. For the assigned values of V^Z, m^Z, m', and m' - m", then, the optimum trajectory is constructed by starting at Z and progressing to a generic point C of the line for which m = m' - m" just as was done previously in Figs. 1 through 4. In order

(footnote continued on next page)

10. Everything that has been said in Article 9 pertains to the case for which $h_C \neq 0$. If it should happen that $h_C = 0$ (or if the similar situation should be met where $\ell_L = 0$), then the Weierstrass function does not tell how long the condition $h = 0$ (or, similarly, the condition $\ell = 0$) is to apply. The preceding illustrative examples show, however, that even in this eventuality there should arise no real difficulty concerning the determination of the character of the sub-regions governing the permissible types of trajectory that are marked off in the (m,V)-plane, provided it is agreed that E is to be taken as non-negative along every optimal path which lies in proximity to the solution under examination. In fact, it was shown in Article 9 how one deals with a similar apparently ambiguous situation. In that illustration there existed a discontinuity in the rate of variation in drag, D_V, at the velocity value of 0.85 V_r, but one was able to determine, satisfactorily, by treating each case individually, just how far along the line representing $V = 0.85\ V_r$ one needed to go. On the other hand, even though another discontinuity in D_V existed at the velocity value of $V = 1.05\ V_r$ it turned out that this line was automatically rejected from being considered as part of any permissible optimal trajectory.

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to find the continuation of the path Z-C one merely starts out now from a point B for which the coordinates are $m^B = m'$ and $V^B = V^C$ while the multiplier λ_3^B is also retained at the value λ_3^C; from this point on one runs along a trajectory scallop which is consistent with the new value of q_1.

In order to furnish more information about how to handle such supposedly indeterminate situations which arise when either h_C or ℓ_L is zero, the additional detailed example will be considered now which illustrates the procedure to be used for climbs executed in minimum time. It is premised that the aircraft weight is going to remain constant, the induced drag is to be zero ($D_L = 0$) and the thrust function is considered to be assigned, or $T = T(z,V)$. In these circumstances then Eqs. (15), (16), and (20) now reduce to

$$\dot{\lambda}_3 + \lambda_2 \sin\vartheta + \lambda_3 N_V/m = 0 \tag{35}$$

$$(\lambda_2 V - \lambda_3 g) \cos\vartheta = 0 \tag{36}$$

and

$$\lambda_2 \dot{z} + \lambda_3 \dot{V} = 1 \tag{37}$$

By use of the relation $\lambda_2 V - \lambda_3 g = 0$ it may be immediately recognized that $\lambda_3 = m/N$, and thus the following singular (see footnote on page 15) solution is obtained for the optimal trajectory:

$$N + VN_V = \frac{V^2}{g} \cdot N_z \tag{38}$$

In addition, as indicated by Eq. (36), it is seen that part of the complete trajectory is composed of sections along which the flight path is vertical; i.e., dives and straight-up zooms may comprise portions of the linked together trajectory.

In regard to the calculation of the Weierstrass function it may be observed that F does not contain $\dot{\vartheta}$ and consequently it will be best to consider the variation $\overset{*}{\Delta}\vartheta$. Thus the Weierstrass condition may be written now as

$$E = (\lambda_3 g - \lambda_2 V)\ \overset{*}{\Delta} \sin\vartheta \tag{39}$$

It is most apropos for bringing out the special features of this problem to consider first the case where there exists a discontinuity, at a certain altitude, in the value of the derivative N_z, so that there will also exist a discontinuity in the value of V there, according to the dictates of Eq. (38). What transpires in the (V,z)-plane in the neighborhood of such an anomalous point is represented in Fig. 5. The jump in the value of N_z may be considered to be so small, when depicted in the scale conveniently employed in this figure, that for all intents and purposes the pertinent curves may be represented by straight lines.

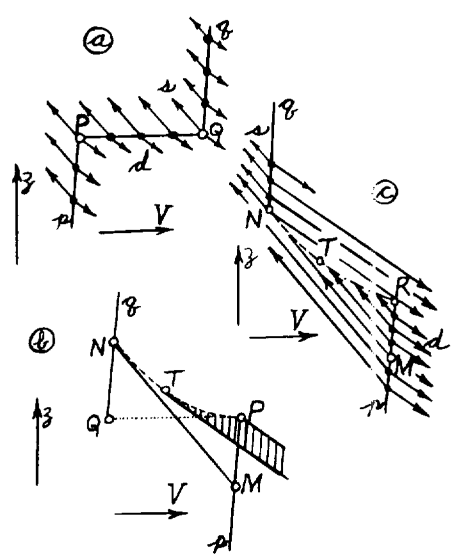

Fig. 5

Trajectory Construction for Vertical Paths Along Which Discontinuities in the Net Thrust Take Place

The situation which arises when N_z diminishes with increasing altitude is readily dealt with, as illustrated in Fig. 5(a). In this case the extremal solution given by Eq. (38) is represented by the line p-P at altitudes below the altitude at which the discontinuity occurs, while it is represented by the line Q-q at altitudes above the discontinuity altitude. These two branches of the extremal solution are joined by the P-Q segment, representing horizontal flight. From any of the points of the composite trajectory p-P-Q-q a zoom "s", or a dive "d", may be initiated.

The graphical representation of what occurs in the case where the discontinuity has the opposite sign, for which, therefore, $V^Q < V^P$, is given in Fig. 5(b). In this figure it is indicated that the local trajectory starts from any point along the extremal p-P and a vertical zoom is initiated there. Then starting with the initial set of values for $\lambda_3 = m/N$ and $\lambda_2 = W/NV$ at such a point on p-P the subsequent values of λ_2 and λ_3 are calculated, using Eqs. (35) and (37), by travelling along the zoom path, until the ratio λ_3/λ_2 again attains the value V/g. When this point is reached the zoom must be abandoned and a dive commenced. This reversal occurs at points along the dashed line which intersects at the point N the branch of the extremal solution denoted by Q-q. In this manner, thus, the zoom trajectory M-N is delineated which forms a bridge between the two branches of the extremal solution.

With this construction carried out adequate description of the significant areas of special interest for variant behavior along sequential parts of the flight path has been supplied, except for the shaded portion of the diagram which is bounded by the "corner" curve labelled P-T and the two boundaries representing dives, one tangent at T and the other starting at P. This region is triply covered, and the minimum time path will have to be selected from among three possible solutions. Thus the completion of the (V,z)-diagram may be accomplished by supplying the scallop of trajectory running from T to R, shown as a dot-dash line in Fig. 5(c), where the point R is located just a little below P of Fig. 5(b) . On the right of this line p-R-T-N-q one finds dives starting out at the points of p-R and of T-N-q, while to the left of this boundary one finds zooms starting out from the segments of line lablled p-R and N-q.

11. The above-given examples demonstrate that even for discontinuous cases, and for singular extremals, the requisite trajectory field-plots may be constructed, which, in a sense, may be treated in such a way as to be considered simply covered. A simply covered field is one, that is, such that for any point in this field of trajectory representations there will exist only one uniquely defined path leading to the assigned terminal-point and fulfilling the condition that E is everywhere non-negative for any variation from the optimal. An exception to this condition will be met, of course, along the singular segments of the extremal represented by straight line portions of the operating curves, for which, in this instance, the value of E is zero for certain virtual deviations but it becomes positive close by to the extremal arcs.

Although the sufficiency theorems of the calculus of variations cannot be applied rigorously in these singular cases, nevertheless it can be demonstrated, as in the above examples, that there do not exist, within the permissible region of operation, any better optimal trajectories leading to the given terminal point than the one constructed by the methods illustrated. For instance, in regard to the problem treated in Articles 8 and 9, it is worth pointing out that if one attempts to traverse a trajectory going from the assigned point A to reach the terminal point Z by following any other path, call it "c", than the one constructed by following the processes illustrated in those Articles, and denoted now as path "e", it will be found that a loss in range ensues, of an amount expressed by $\int_A^Z E\,dt$. This integral is computed by following along the alternate path "c" and using in the summation process the local values found for λ_3, m, and q_1, with the further understandings that the condition $\Delta^*L = 0$ holds and that the Δ^*C and Δ^*N deviations are to be computed as the differences between the values pertaining to the alternate path "c" and the locally optimal ones.* This integration will turn out to give a positive result for any alternative path differing from the optimal one.†

Naturally, the mode of approach† and proof, being advocated here, requires that a great deal of exploration and probing

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* The expression for E will be indeterminate in the region of the trajectory-field covered by the lines V = constant, because in this region it is true that $q_1 = 0$ while $\Delta^*C = \infty$. In this eventuality, however, it may be readily shown that in this region one may write

$$E = -V - \lambda_3 \dot{V}$$

where $\dot{V}$ is the value of dV/dt that applies along the alternative path "c", and hence it is not difficult to prove that in this case also it turns out that $E \geq 0$.

† It is being tacitly assumed here that the problem meant is one in which the terminal point is assigned.

of permissible boundaries and characteristics of the pertinent trajectory-field be carried out. At least, even if one restricts the investigation to the determination merely of a relative minimum, still the region close to the solution being sought must be thoroughly mapped out and examined in detail. Such backing-and-filling type of investigations and exploratory probings will increase greatly in complexity, of course, if the number of variables which come into play are allowed to increase. On the other hand, it may be remarked that even in the case of actual variational problems where, as a rule, both the terminal and initial points are assigned or are subject to prescribed conditions, the optimal trajectory is also obtained by trial and error. The tentative solutions used in such trial-balloon operations also serve to test out and define the trajectory-field under examination.

Translated by R. H. Cramer
Applied Physics Laboratory
The Johns Hopkins University
Silver Springs, Maryland

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